

## LEFT-SEPARATED TOPOLOGICAL SPACES

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ABSTRACT. A well ordering  $\prec$  of a topological space  $X$  is *left-separating* if  $\{x' \in X : x' \prec x\}$  is closed in  $X$  for any  $x \in X$ . A space is *left-separated* if it has a left-separating well-ordering. The *left-separating type*  $\text{ord}_\ell(X)$  of a left-separated space  $X$  is the minimum of the order types of the left-separating well orderings of  $X$ .

We prove that

- (1) for each uncountable regular cardinal  $\kappa$  and for each ordinal  $\alpha < \kappa^+$  there is a  $T_2$  space  $X$  such that  $\alpha \leq \text{ord}_\ell(X) < \kappa^+$ ;
- (2) it is consistent that for each uncountable regular cardinal  $\kappa$  and for each ordinal  $\alpha < \kappa^+$  there is a 0-dimensional  $T_2$  space  $X$  such that  $\alpha \leq \text{ord}_\ell(X) < \kappa^+$ ;
- (3) if  $\kappa$  and  $\lambda$  are infinite cardinals such that  $\kappa = \lambda^\omega = 2^\lambda$  then for all  $\alpha < \kappa^+$  there is a 0-dimensional  $T_2$  space  $X$  such that  $\alpha \leq \text{ord}_\ell(X) < \kappa^+$ .

The union of two left-separated spaces is not necessarily left-separated. We show, however, that if  $X$  is a countably tight space,  $X = Y \cup Z$ ,  $\text{ord}_\ell(Y) < \omega_1 \cdot \omega$  and  $\text{ord}_\ell(Z) < \omega_1 \cdot \omega$ , then  $X$  is also left-separated and

$$\text{ord}_\ell(X) \leq \text{ord}_\ell(Y) + \text{ord}_\ell(Z).$$

We prove that it is consistent that there is a first countable, 0-dimensional space  $X$ , which is not left-separated, but there is a c.c.c poset  $Q$  such that

$$V^Q \models \text{ord}_\ell(X) = \omega_1 \cdot \omega.$$

However, if  $X$  is a topological space and  $Q$  is a c.c.c poset such that

$$V^Q \models \text{ord}_\ell(X) < \omega_1 \cdot \omega,$$

then  $X$  is left-separated even in  $V$ .

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## 1. INTRODUCTION

The notion of left- and right-separated spaces was introduced by Hajnal and Juhász [3].

A well ordering  $\prec$  of a topological space  $X$  is *left-separating* (*right-separating*) if  $\{x \in X : x \prec y\}$  is closed (respectively, open) in  $X$  for any  $y \in X$ . A space is *left-separated* (*right-separated*) if it has a left-separating (respectively, right-separating) well-ordering. The *left separating type*  $\text{ord}_\ell(X)$  of a left-separated space  $X$  is the minimum of the order types of the left-separating well orders of  $X$ . The *right separating type*  $\text{ord}_r(X)$  of a right-separated space  $X$  is defined analogously: it is the minimum of the order types of the right-separating well orders of  $X$ . We write  $\text{ord}_\ell(X) = \infty$  ( $\text{ord}_r(X) = \infty$ ) if  $X$  is not left-separated (respectively, not right-separated).

A space,  $X$ , is *scattered* if every non-empty subspace of  $X$  has an isolated point. The Cantor-Bendixson height of a scattered space  $X$  will be denoted by  $\text{ht}(X)$ .

A space is scattered if and only if it is right-separated. Since  $\text{ht}(X) \leq \text{ord}_r(X)$  holds for scattered spaces, and  $\text{ht}(\omega^\alpha) = \alpha$  for ordinals  $\alpha > 0$ , where  $\omega^\alpha$  denotes ordinal exponentiation, then for each infinite ordinal  $\alpha$  we have  $|\omega^\alpha| = |\alpha|$  and  $\text{ord}_r(\omega^\alpha) \geq \alpha$ . So the right-separating types of spaces of cardinality  $\kappa$  are unbounded in  $\kappa^+$ .

In section 2, we discuss the same problem for left-separated spaces:

*Let  $\kappa$  be an infinite cardinal and  $\alpha < \kappa^+$ . Is there a left-separated regular (or Hausdorff) space  $X$  of size  $\kappa$  with  $\text{ord}_\ell(X) > \alpha$  (or  $\text{ord}_\ell(X) = \alpha$ )?*

The answer is clearly **no** for  $\kappa = \omega$ : every countable  $T_1$  space is left-separated in type  $\omega$ . In theorems 1.1 and 1.3 we give a partial answer for cases where  $\kappa > \omega$ .

**Theorem 1.1.** *If  $\kappa$  is a regular, uncountable cardinal, and  $\kappa \leq \alpha < \kappa^+$  is an ordinal, then there is a first countable, scattered  $T_2$  space  $X = \langle X, \tau \rangle$  such that  $\alpha + 1 \leq \text{ord}_\ell(X) \leq \kappa \cdot (\alpha + 1)$ .*

To find regular spaces with large left-separating type we need extra assumptions. First of all, we should introduce the principle  $t_{\text{stat}}(\kappa)$ , which is a weakening of  $\clubsuit(E_\omega^\kappa)$ . Let us remark that principle (t) was introduced by Juhász in [4].

**Definition 1.2.** Let  $\kappa$  be an infinite cardinal. Given a stationary subset  $S \subset E_\omega^\kappa$  we say that a sequence  $\langle A_\alpha : \alpha \in S \rangle$  is a *t-sequence* if

- (1)  $A_\alpha$  is a countable subset of  $\alpha$  converging to  $\alpha$  for all  $\alpha \in S$ ,

(2) For each stationary subset  $T \subset \kappa$

the set  $\{\alpha \in S : A_\alpha \cap T \text{ is infinite}\}$  is stationary in  $\kappa$ .

Principle  $t_{stat}(\kappa)$  holds if there is a  $t$ -sequence for each  $S \subseteq^{stat} E_\omega^\kappa$ .

**Theorem 1.3.** *Let  $\kappa$  be an uncountable cardinal. If*

*(a)  $\kappa$  is a regular cardinal and  $t_{stat}(\kappa)$  holds,*

*or*

*(b)  $\kappa = \lambda^\omega = 2^\lambda$  for some  $\lambda < \kappa$ ,*

*then for each ordinal  $\kappa \leq \alpha < \kappa^+$ , there is a locally compact, locally countable, first countable, scattered, 0-dimensional  $T_2$  space  $X = \langle X, \tau \rangle$  such that  $\alpha + 1 \leq \text{ord}_\ell(X) \leq \kappa \cdot (\alpha + 1)$ .*

The following question remained open.

**Problem 1.4.** *Is it true, in ZFC, that for each uncountable cardinal  $\kappa$  and for each ordinal  $\alpha < \kappa$  there is a 0-dimensional,  $T_2$  space  $X$  with  $\alpha < \text{ord}_\ell(X) < \kappa^+$ ?*

The union of two scattered space is clearly scattered, so the class of right-separated spaces is closed under finite union. In section 3 we investigate the same question for left-separated spaces. First we show that *there are two dense left-separated subsets  $A$  and  $B$  of  $2^c$  such that  $A \cup B$  is not left-separated* (see Example 3.2). On the other hand, we also get some positive results.

**Theorem 1.5.** *If  $A$  and  $B$  are left-separated spaces,  $t(A \cup B) = \omega$  and  $\text{ord}_\ell(A) + \text{ord}_\ell(B) < \omega_1 \cdot \omega$ , then  $A \cup B$  is left-separated and  $\text{ord}_\ell(A \cup B) < \omega_1 \cdot \omega$ .*

Properties “left-separated and “right-separated” are upward absolute because a left-separating (respectively, right-separating) well-order remains left-separating (respectively, right-separating) in any extension of the ground model.

Property “right-separated” is also downward absolute because if a space is scattered in any extension extension that it is also scattered in the ground model. So the property “right-separated” is absolute.

What about property “left-separated”? Since a countable  $T_1$  space is automatically left-separated, we consider only cardinal preserving extensions. Moreover, a subspace  $S \subset \omega_1$  is left-separated if and only if it is stationary, we consider only stationary preserving forcing extensions.

In section 4, we investigate if c.c.c generic extensions preserve “not left-separated” property. We prove the following result.

**Theorem 1.6.** (1) *If  $X$  is a topological space,  $Q$  is a c.c.c poset such that*

$$V^Q \models \text{ord}_\ell(X) < \omega_1 \cdot \omega,$$

*then  $X$  is left-separated even in  $V$ , moreover*

$$\text{ord}_\ell(X)^V = \text{ord}_\ell(X)^{V^Q}.$$

(2) *It is consistent that there is a 0-dimensional, first countable topological space  $X$  such that*

$$\text{ord}_\ell(X) = \infty,$$

*but*

$$V^Q \models \text{ord}_\ell(X) = \omega_1 \cdot \omega$$

*for some c.c.c poset  $Q$ .*

The following questions remained open.

**Problem 1.7.** *Is there, in ZFC, a 0-dimensional, first countable topological space  $X$  which is not left-separated, but which becomes left-separated in some c.c.c generic extension?*

**Problem 1.8.** *Is it consistent that there is a left-separated space  $X$  such that*

$$\text{ord}_\ell(X)^{V^Q} < \text{ord}_\ell(X)^V$$

*for some c.c.c.(proper) poset  $Q$ ?*

### Notation.

If  $P$  is a property, we write  $X \subset^P Y$  to mean that “ $X \subset Y$  and  $X$  has property  $P$ ”. In particular,

- if  $\kappa$  is a cardinal and  $S, T \subset \kappa$ , then we write  $S \subset^{stat} T$  if  $S \subset T$  and  $S$  is stationary in  $\kappa$ .
- if  $X$  is a topological space and  $A$  is a set, then we write  $A \subset^{closed} X$  if  $A \subset X$  and  $A$  is closed in  $X$ .

## 2. THE VALUES OF THE $\text{ord}_\ell$ FUNCTION

Given a cardinal  $\kappa$  we say that  $T \subset^{<\omega} \kappa$  is a *well-founded tree* if  $T$  is closed under initial segments, and the poset  $\mathcal{T} = \langle T, \subset \rangle$  is a tree without infinite branches.

If  $T \subset \kappa^{<\omega}$  is a well-founded tree, , then we can define the rank function

$$\text{rank}_{\mathcal{T}} : T \rightarrow \mathbf{On}$$

by the recursive formula

$$\text{rank}_{\mathcal{T}}(s) = \sup\{\text{rank}_{\mathcal{T}}(t) + 1 : s \subsetneq t \in T\}.$$

Hence the leaves have rank 0. The rank of  $\mathcal{T}$  is defined by the formula

$$\text{rank}(\mathcal{T}) = \sup\{\text{rank}_{\mathcal{T}}(t) + 1 : t \in T\} = \text{rank}_{\mathcal{T}}(\emptyset) + 1.$$

To prove Theorems 1.1 and 1.3(a) we need the following result.

**Theorem 2.1.** *Assume that*

(S1)  $\kappa$  is an uncountable regular cardinal,

(S2)  $T \subset \kappa^{<\omega}$  is a well-founded tree.

(S3)  $\{X_t : t \in T\}$  is a family of pairwise disjoint, stationary subsets of  $\kappa$ .

(S4)  $\tau$  is a topology on  $X = \bigcup_{t \in T} X_t$  which refines the order topology.

(a) If

(S5) every  $x \in X_t$  has a neighborhood  $U(x)$  such that

$$U(x) \setminus \{x\} \subset \bigcup \{X_s : s \subsetneq t\}$$

then  $X$  is left-separated and

$$\text{ord}_{\ell}(\langle X, \tau \rangle) \leq \kappa \cdot \text{rank}(\mathcal{T}).$$

(b) If

(S6) for each  $s \subset t \in T$  and  $S \subset^{stat} X_s$  we have  $\overline{S}^{\tau} \cap X_t \subset^{stat} \kappa$ .

then  $X$

$$\text{rank}(\mathcal{T}) \leq \text{ord}_{\ell}(\langle X, \tau \rangle).$$

*Proof of Theorem 2.1.* Write  $\alpha = \text{rank}(\mathcal{T})$ .

Consider a bijection  $f : X \rightarrow \kappa \cdot \alpha$  such that for all  $\zeta < \alpha$

$$\{f(x) : x \in \bigcup_{\text{rank}_{\mathcal{T}}(t)=\zeta} X_t\} = \kappa \cdot \{\zeta\}.$$

Then  $f^{-1}\delta \subset^{closed} X$  for all  $\delta < \kappa \cdot \alpha$  by (S5) because  $s \subsetneq t$  implies  $\text{rank}_{\mathcal{T}}(s) > \text{rank}_{\mathcal{T}}(t)$ . So  $\text{ord}_{\ell}(X) \leq \kappa \cdot \alpha$ .

(b)

Assume that  $\text{ord}_{\ell}(X) = \beta$ , and fix a bijection

$$f : X \rightarrow \beta$$

such that for each  $\beta' < \beta$

$$f^{-1}\beta' \subset^{closed} X.$$

Define a function  $\rho : T \rightarrow \mathbf{On}$  as follows:

$$\rho(t) = \min\{\beta' \leq \beta : (f^{-1}\beta' \cap X_t) \subset^{stat} \kappa\}.$$

Since the non-stationary ideal on  $\kappa$  is  $\kappa$ -complete we have  $cf(\rho(t)) = \kappa$  for all  $t \in T$ .

**Claim 2.1.1.** *If  $s \subsetneq t$ , then  $\rho(t) < \rho(s)$ .*

*Proof of the Claim.* Write  $\delta = \rho(s)$ .

Since  $(f^{-1}\delta \cap X_s) \subset^{\text{stat}} \kappa$  it follows

$$\overline{(f^{-1}\delta \cap X_s)}^\tau \cap X_t \subset^{\text{stat}} \kappa$$

by (S6).

Since  $f^{-1}\delta$  is  $\tau$ -closed,

$$\overline{(f^{-1}\delta \cap X_s)}^\tau \subset f^{-1}\delta.$$

Thus

$$X_t \cap f^{-1}\delta \supset \overline{(f^{-1}\delta \cap X_s)}^\tau \cap X_t.$$

Hence  $X_t \cap f^{-1}\delta \subset^{\text{stat}} \kappa$ , and so  $\rho(t) \leq \delta$ .

Assume on the contrary that  $\rho(s) = \rho(t) = \delta$ , and let

$$Y_s = X_s \cap f^{-1}\delta \subset^{\text{stat}} \kappa$$

and

$$Y_t = X_t \cap \overline{Y_s} \subset^{\text{stat}} \kappa.$$

Fix a club set  $C = \{\gamma_\nu : \nu < \kappa\}$  in  $\delta$ , and consider

$$D = \{\nu < \kappa : f[\nu] \cap \delta \subset \gamma_\nu\}.$$

If  $\nu \in Y_t \cap D$ , then  $\nu \in \overline{Y_s \cap \nu}^\tau$  because  $\nu \in \overline{Y_s}$  and  $\tau$  refines the order topology on  $\kappa$ .

But  $f[Y_s \cap \nu] \subset f[\nu] \cap \delta \subset \gamma_\nu$ , and  $f^{-1}\gamma_\nu \subset^{\text{closed}} X$ , so  $\nu \in f^{-1}\gamma_\nu$ , i.e.  $f(\nu) < \gamma_\nu$ .

Since  $C$  is club, and  $Y_t \cap D$  is stationary there is  $S \subset^{\text{stat}} Y_t \cap D$  and  $\mu < \kappa$  such that  $f(\nu) < \gamma_\mu$  for all  $\nu \in S$ .

Thus  $S \subset f^{-1}\gamma_\mu$ , and so  $\rho(t) \leq \gamma_\mu < \delta$ . Contradiction,  $\rho(t) = \rho(s)$  is not possible, and so the claim holds.  $\square$

By transfinite induction, the Claim implies that  $\text{rank}_{\mathcal{T}}(s) \leq \rho(s)$  for all  $s \in T$ , and so  $\alpha \leq \text{ord}_\ell(\langle X, \tau \rangle)$ .  $\square$

*Proof of Theorem 1.1.* Let  $T \subset \kappa^{<\omega}$  be a well-founded tree with  $\text{rank}(\mathcal{T}) = \alpha + 1$ .

If  $t \in T$  is not the root of the tree then denote  $\text{pred}(t)$  the predecessor of  $t$  in  $\mathcal{T}$ .

Let  $\{R_t : t \in T\}$  be a family of pairwise disjoint stationary subsets of  $E_\kappa^\omega$ . By induction on  $|t|$  define stationary sets  $X_t \subset R_t$  as follows:

$$X_t = R_t \cap \bigcap \{X'_s : s \subsetneq t\},$$

and let  $X = \bigcup_{t \in T} X_t$ .

Write

$$X_{\prec t} = \bigcup \{X_s : s \subsetneq t\}.$$

We define the topology  $\tau$  on  $X$  as follows.

For  $t \in T$  and  $\gamma \in X_t$ , for all  $\beta < \gamma$  let

$$B_\gamma(\beta) = \{\gamma\} \cup ((\beta, \gamma) \cap X_{\prec t}).$$

The neighborhood base of  $\gamma$  in  $X$  will be

$$\{B_\gamma(\beta) : \beta < \gamma\}.$$

We are to show that  $\kappa$ ,  $T$ ,  $\{X_t : t \in T\}$  and  $\langle X, \tau \rangle$  satisfy (S1)–(S6).

The topology  $\tau$  is finer than the usual order topology of  $\kappa$ , so  $\langle X, \tau \rangle$  is a scattered  $T_2$  space. Since  $cf(\alpha) = \omega$  for all  $\alpha \in X$ , the space is first countable.

(S1)–(S5) are clear from the construction.

To check (S6), assume that  $s \subsetneq t$  and  $S \subset^{\text{stat}} X_s$ . Then

$$\overline{S} \cap X_t = \overline{S}^\tau \cap X_t$$

by the definition of  $\tau$ . So  $\overline{S}^\tau \cap X_t$  is stationary because  $\overline{S} \subset^{\text{club}} \kappa$ .

So we can apply Theorem 2.1 to obtain  $\alpha + 1 \leq \text{ord}_\ell(X) \leq \kappa \cdot (\alpha + 1)$ .  $\square$

*Proof of Theorem 1.3(a).* Let  $T \subset \kappa^{<\omega}$  be a well-founded tree with  $\text{rank}(\mathcal{T}) = \alpha + 1$ .

If  $t \in T$  is not the root of the tree then denote  $\text{pred}(t)$  the predecessor of  $t$  in  $\mathcal{T}$ .

Let  $\{R_t : t \in T\}$  be a family of pairwise disjoint stationary subsets of  $E_\kappa^\omega$ . By induction on  $|t|$  define stationary sets  $X_t \subset R_t$  as follows:

$$X_t = R_t \cap \bigcap \{X'_s : s \subsetneq t\},$$

and let  $X = \bigcup_{t \in T} X_t$ .

For each  $t \in T$ , let  $\langle D_\gamma : \gamma \in X_t \rangle$  be a  $t$ -sequence.

We define, by transfinite induction, a sequence

$$\langle \langle Y_\gamma, \tau_\gamma \rangle : \gamma \leq \kappa \rangle$$

of topological spaces, where  $Y_\gamma = X \cap \gamma$ , such that

- (i)  $\langle Y_\gamma, \tau_\gamma \rangle$  is locally compact and locally countable,
- (ii)  $\tau_\gamma$  refines the usual order topology on  $X \cap \gamma$ ,

- (iii)  $\tau_\beta = \tau_\gamma \cap \mathcal{P}(X \cap \beta)$  for  $\beta < \gamma$ .
- (iv) for all  $\xi \in X_t \cap \gamma$  there is a neighborhood  $U(\xi) \in \tau_\gamma$  such that

$$U(\xi) \setminus \{\xi\} \subset \bigcup \{X_s : s \subsetneq t\}.$$

**Case 1:**  $\gamma = 0$ .

Then  $\tau_\emptyset = \{\emptyset\}$ .

**Case 2:**  $\gamma$  is a limit ordinal.

Let the topology  $\tau_\gamma$  be generated by the family  $\bigcup_{\beta < \gamma} \tau_\beta$ .

**Case 3:**  $\gamma = \beta + 1$

Let  $t \in T$  so that  $\beta \in X_t$ . If  $\beta \notin X$ , then let  $\tau_\gamma = \tau_\beta$ .

If  $\beta \in X_{\text{root}(T)}$  or  $D_\beta \cap X_{\text{pred}(t)}$  is finite, then declare  $\beta$  to be isolated in  $\tau_\gamma$ .

Assume now that  $\beta \in X_t$ ,  $s = \text{pred}(t)$  and  $E_\beta = D_\beta \cap X_s$  is infinite. Let  $\langle \beta_n : n \in \omega \rangle$  be the strictly increasing enumeration of  $E_\beta$ , and pick pairwise disjoint compact open neighborhoods  $U_n$  of  $\beta_n$  in  $\tau_\beta$  such that

( $\star$ )  $U_n \setminus \{\beta_n\} \subset (\beta_{n-1}, \beta_n) \cap (\bigcup \{X_r : r \prec s\})$ .

Let

$$B_n(\beta) = \{\beta\} \cup \bigcup_{m \geq n} U_m.$$

for  $n \in \omega$ . The topology  $\tau_\gamma$  is generated by the family

$$\tau_\beta \cup \{B_n(\beta) : n \in \omega\}.$$

It is clear from the construction that the sets  $B_n(\beta)$  are compact, and so the spaces  $\langle Y_\gamma, \tau_\gamma \rangle$  are locally compact and clearly locally countable.

So the space  $X = \langle X, \tau_\kappa \rangle$  is scattered, locally compact and locally countable. So it is also 0-dimensional.

We are to show that  $\kappa, T, \{X_t : t \in T\}$  and  $\langle X, \tau_\kappa \rangle$  satisfy conditions (S1)–(S6) of Theorem 2.1.

(S1)–(S5) are clear from the construction.

To check (S5), assume that  $s \subsetneq t$  and  $S \subset^{\text{stat}} X_s$ . We can also assume that  $s = \text{pred}(t)$ .

Since  $\{D_\nu : \nu \in X_t\}$  is a  $t$ -sequence

$$\{\nu \in X_t : |D_\nu \cap S| = \omega\} \subset^{\text{stat}} X_t,$$

and

$$\overline{S}^{\tau_\kappa} \cap X_t \supset \{\nu \in X_t : |D_\nu \cap S| = \omega\}.$$

by the definition of  $\tau_\kappa$ . So (S6) holds.

So we can apply Theorem 2.1 to obtain  $(\alpha + 1) \leq \text{ord}_\ell(X) \leq \kappa \cdot (\alpha + 1)$ .  $\square$



To prove Theorem 1.3(b) we need the following result. To formulate it we recall some notions and notations from [1]. The discrete topological space on a set  $A$  will be denoted by  $D(A)$ . The *Baire space of weight  $\lambda$* ,  $B(\lambda)$ , is the metric space  $D(\lambda)^\omega$ , see [1, Example 4.2.12]. The metric topology of  $B(\lambda)$  will be denoted by  $\rho_\lambda$ .

**Theorem 2.2.** *Assume that  $\kappa$  and  $\lambda$  are infinite cardinals,  $\kappa = \lambda^\omega = 2^\lambda$ , and*

- (C1)  $T \subset \kappa^{<\omega}$  a well-founded tree,
- (C2)  $\{X_t : t \in T\}$  is a family of pairwise disjoint elements of  $[B(\lambda)]^\kappa$ ,
- (C3)  $\tau$  is a topology on  $X = \bigcup_{t \in T} X_t$  which refines the Baire topology  $\rho_\lambda$ .

(a) *If*

- (C4) *every  $x \in X_t$  has a neighborhood  $U(x)$  such that*

$$U(x) \setminus \{x\} \subset \bigcup \{X_s : s \subsetneq t\}$$

*then  $X$  is left-separated and*

$$\text{ord}_\ell(\langle X, \tau \rangle) \leq \kappa \cdot \text{rank}(\mathcal{T}).$$

(b) *If*

- (C5) *for each  $s \subsetneq t \in T$  and  $S \subset X_s$ , if  $|\overline{S}^{\rho_\lambda}| = \kappa$  then  $|\overline{S}^\tau \cap X_t| = \kappa$ , then*

$$\text{rank}(\mathcal{T}) \leq \text{ord}_\ell(\langle X, \tau \rangle).$$

*Proof of Theorem 2.2.* (a) Write  $\alpha = \text{rank}(\mathcal{T})$ . Consider a bijection  $f : X \rightarrow \kappa \cdot \alpha$  such that for all  $\beta < \alpha$

$$\{f(x) : x \in \bigcup_{\text{rank}_{\mathcal{T}}(t)=\beta} X_t\} = \kappa \cdot \{\beta\}.$$

Then  $f^{-1}\delta \subset^{\text{closed}} X$  for all  $\delta < \kappa \cdot \alpha$  by (C4). So  $\text{ord}_\ell(X) \leq \kappa \cdot \alpha$ .

(b) Assume that  $\text{ord}_\ell(X) = \beta$ , and so we can fix a bijection

$$f : X \rightarrow \beta$$

such that  $f^{-1}\beta' \subset^{\text{closed}} X$  for each  $\beta' < \beta$ .

Let

$$\mathcal{A} = \{A \in [B(\lambda)]^\lambda : |\overline{A}^{\rho_\lambda}| = \kappa\}.$$

Define  $\rho : T \rightarrow \mathbf{On}$  as follows:

$$\rho(t) = \min\{\beta' \leq \beta : f^{-1}\beta' \cap X_t \text{ contains an element of } \mathcal{A}\}.$$

Since  $w(B(\lambda)) = \lambda$  the function  $\rho$  is defined everywhere.

**Claim 2.2.1.** *If  $s = \text{pred}(t)$ , then  $\rho(t) < \rho(s) = \delta$ .*

*Proof of the Claim.* Assume that  $A \in \mathcal{A} \cap \mathcal{P}(f^{-1}\delta \cap X_s)$ .

By the minimality of  $\delta$  we have  $\text{cf}(\delta) \leq \lambda$ .

Then  $|\overline{A}^{\rho_\lambda}| = \kappa$ , and so  $|\overline{A}^\tau \cap X_t| = \kappa$  by (C5) and  $\overline{A}^\tau \subset f^{-1}\delta$  by left-separability.

So  $B = \overline{A}^\tau \cap X_t \subset f^{-1}\delta$  has cardinality  $\kappa$ .

Since  $\text{cf}(\kappa) = \text{cf}(2^\lambda) > \lambda$ , there is  $\gamma < \delta$  such that  $|B \cap f^{-1}\gamma| = \kappa$  and so  $B \cap f^{-1}\gamma$  contains a dense subset from  $\mathcal{A}$ . So  $\rho(t) \leq \gamma$ .  $\square$

The Claim implies that  $\text{rank}(s) \leq \rho(s)$  for all  $s \in T$ , and so  $\alpha \leq \text{ord}_\ell(X)$ .  $\square$

*Proof of Theorem 1.3(b).* Let  $T \subset (2^\omega)^{<\omega}$  be a well-founded tree such that  $\text{rank}(\mathcal{T}) = \alpha + 1$ .

If  $t \in T$  is not the root of the tree then denote  $\text{pred}(t)$  the predecessor of  $t$  in  $\mathcal{T}$ .

Let

$$\mathcal{D} = \{D \in [\mathbf{B}(\lambda)]^\lambda : |\overline{D}^{\rho_\lambda}| = \kappa\},$$

and fix an enumeration  $\{D_\nu : \nu < 2^\omega\}$  of  $\mathcal{D}$ .

Let  $\{\langle t_i, \nu_i, \zeta_i \rangle : i < 2^\omega\} = T \times \kappa \times \kappa$ . We define, by transfinite induction, an increasing sequence  $\langle \langle X^\gamma, \tau_\gamma \rangle : \gamma \leq \kappa \rangle$  of topological space such that  $X^\gamma \subset^{\text{open}} X^\beta$  for  $\gamma < \beta$ , moreover we also define sets  $\{\langle X_t^\gamma : t \in T \rangle : \gamma \leq \kappa\}$  with  $X^\gamma = \bigcup_{t \in T} X_t^\gamma$  as follows.

**Case 1:**  $\gamma = 0$ .

Let  $X^0 = \emptyset$  and  $X_t^0 = \emptyset$  for  $t \in T$ .

**Case 2:**  $\gamma$  is a limit ordinal.

Let  $X_t^\gamma = \bigcup_{\beta < \gamma} X_t^\beta$  and  $X^\gamma = \bigcup_{\beta < \gamma} X^\beta$ , and the topology  $\tau_\gamma$  is generated by the family  $\bigcup_{\beta < \gamma} \tau_\beta$ .

**Case 3:**  $\gamma = \beta + 1$ .

Consider the triple  $\langle t_\beta, \nu_\beta, \zeta_\beta \rangle$ .

If  $t_\beta$  is the root of the tree: just pick a new element  $x_\beta \in \mathbf{B}(\lambda) \setminus X^\beta$ , let  $X_{t_\beta}^\gamma = X_{t_\beta}^\beta \cup \{x_\beta\}$ , let  $X_t^\gamma = X_t^\beta$  for  $t \in T \setminus \{t_\beta\}$ , and declare that  $x_\beta$  to be isolated in  $\tau_\gamma$ .

Assume now that  $t_\beta$  is not the root of the tree. Let  $s = \text{pred}(t_\beta)$ .

If  $D_{\nu_\beta} \not\subset X_s^\beta$ , then do nothing: let  $X^\gamma = X^\beta$  and  $X_t^\gamma = X_t^\beta$  for all  $t \in T$ .

Assume finally that  $D_{\nu_\beta} \subset X_s^\beta$ . Then pick  $x_\beta \in \overline{D_{\nu_\beta}}^{\rho_\lambda} \setminus X_\beta$ , then let  $X^\gamma = X^\beta \cup \{x_\beta\}$ ,  $X_{t_\beta}^\gamma = X_{t_\beta}^\beta \cup \{x_\beta\}$ , and  $X_t^\gamma = X_t^\beta$  for  $t \in T \setminus \{t_\beta\}$ .

To define the topology  $\tau_\gamma \supset \tau_\beta$  pick a sequence  $\{x_{\beta,n}\}_{n \in \omega} \subset D_{\nu_\beta}$  converging to  $x_\beta$  in the metric topology  $\mu\rho_\lambda$ , and pick pairwise disjoint compact open neighborhoods  $U_n$  of  $x_{\beta,n}$  in  $\tau_\beta$  such that

- (a)  $U_n \setminus \{x_{\beta,n}\} \subset \bigcup \{X_r^\beta : r \prec s\}$ , and
- (b) the sequence of sets  $\{U_n : n \in \omega\}$ , converges to  $x_\beta$  in the metric topology  $\rho_\lambda$ .

Let

$$B_n(x_\beta) = \{x_\beta\} \cup \bigcup_{m \geq n} U_m.$$

for  $n \in \omega$ . The topology  $\tau_\gamma$  is generated by the

$$\tau_\beta \cup \{B_n(x_\beta) : n \in \omega\}.$$

It is clear from the construction that the spaces  $\langle X^\gamma, \tau_\gamma \rangle$  are locally compact and locally countable.

It is clear from the construction that  $\kappa$ ,  $T$ ,  $\{X_t^{2^\omega} : t \in T\}$  and  $X = X^\kappa$  satisfy conditions (C1)-(C5) from Theorem 2.2.

So we can apply Theorem 2.2 to obtain  $\alpha + 1 \leq \text{ord}_\ell(X) \leq \kappa \cdot (\alpha + 1)$ .  $\square$

### 3. ADDITIVITY

The union of two scattered spaces is clearly scattered. What about left-separated spaces?

We will present an example showing that the union of two left-separated spaces is not necessarily left-separated. We need the following lemma.

**Lemma 3.1.** *If  $X$  is a topological space,  $X = A \cup B$ , both  $A$  and  $B$  are dense in  $X$ , and*

- (\*)  *$|A|$ -many nowhere dense subsets of  $B$  can not cover a nonempty open subset of  $B$ ,*

*then  $X$  is not left-separated.*

*Proof.* Assume on the contrary that  $X$  is left-separated witnessed by the enumeration  $X = \{x_\xi : \xi < \mu\}$ , i.e.  $\{x_\zeta : \zeta < \xi\} \subset^{closed} X$  for each  $\xi < \mu$ .

Let

$$\nu = \min\{\mu' \leq \mu : U = \overline{\text{int}\{x_\xi : \xi < \mu'\}} \neq \emptyset\}.$$

Since  $\{x_\xi : \xi < \nu\}$  is closed in  $X$ , it follows, that  $A \cap U \subset \{x_\xi : \xi < \nu\}$ . Since  $A \cap U$  is dense in  $U$ , by the minimality of  $\nu$  we have  $\text{cf } \nu \leq |A|$ . Let  $\{\nu_n : n < \text{cf}(\nu)\}$  be cofinal in  $\nu$ .

Then  $\{a_\xi : \xi < \nu_n\} \cap B$  is nowhere dense for  $n < \text{cf}(\nu)$ , so by  $(*)$  the set

$$\{x_\xi : \xi < \nu\} \cap B = \bigcup_{n < \text{cf}(\nu)} (\{x_\xi : \xi < \nu_n\} \cap B)$$

can not cover a open subset of  $B$ .

However the set  $\{x_\xi : \xi < \nu\}$  is closed in  $X$ , which implies that  $B \cap U \subset \{x_\xi : \xi < \nu\}$ , which is a contradiction.  $\square$

The next example was constructed by Juhász, Soukup and Szentmiklóssy.

**Example 3.2.** *There are two dense left-separated subsets  $A$  and  $B$  of  $2^\mathfrak{c}$  such that  $A \cup B$  is not left-separated.*

*Proof.* Let  $A$  be a countable dense subsets of  $2^\mathfrak{c}$ . Let  $B$  be a  $G_\delta$ -dense, left-separated subset of  $2^\mathfrak{c}$  with  $|B| = \mathfrak{c}$ .

Then  $\text{ord}_\ell(A) = \omega$  and  $\text{ord}_\ell(B) = \mathfrak{c}$ . Since  $B$  is  $G_\delta$ -dense, countably many nowhere dense sets can not cover a nonempty open subset of  $B$ . So, by lemma 3.1, the space  $X = A \cup B$  is not left-separated.  $\square$

**Theorem 3.3.** *Let  $(X_i)_{i \in n}$  be a finite family of left-separated spaces such that for each  $i \in n$ ,  $\text{ord}_\ell(X_i) = \omega_1$ . In addition, suppose  $t(X) = \omega$  where  $X = \bigcup_{i \in n} X_i$ . Then  $X$  is left-separated and  $\text{ord}_\ell(X) \leq \omega_1 \cdot n$ .*

*Proof.* Let  $(X_i)_{i \in n}$  be as stated in the hypothesis of the theorem. Let  $X = \bigcup_{i \in n} X_i$ . For each  $i \in n$  and for each  $A \in [n]^i$ , define

$$Z_A = \left( \bigcap_{j \notin A} \overline{X_j} \right) \setminus \left( \bigcup_{j \in A} \overline{X_j} \right).$$

For each  $i \in n$ , define  $Y_i = \bigcup_{A \in [n]^i} Z_A$ .

**Claim 3.3.1.** *If  $A, B \in [n]^{<n}$  and  $B \setminus A \neq \emptyset$  then  $\overline{Z_A} \cap Z_B = \emptyset$ .*

**Claim 3.3.2.** *For each  $i \in n$ ,  $\bigcup_{j \leq i} Y_j$  is closed in  $X$ , and  $\{Y_i : i \in n\}$  forms a partition of  $X$ .*

**Claim 3.3.3.** *For each  $i \in n$ ,  $Y_i$  is left-separated and  $\text{ord}_\ell(Y_i) \leq \omega_1$ .*

These three claims prove our theorem. Indeed, let  $\preceq_i$  be a left-separating well-ordering of  $Y_i$  in type  $\leq \omega_1$ . Define the well-ordering  $\preceq$  of  $X$  as follows:

$$y \preceq z \text{ iff } (y, z \in Y_i \text{ and } y \preceq_i z) \text{ or } (y \in Y_i \text{ and } z \in \bigcup_{j > i} Y_j).$$

Since the  $\preceq_i$  are left-separating and Claim 3.3.2 holds, the initial segments of  $(X, \preceq)$  are closed. Moreover,  $tp(X, \preceq) \leq \omega_1 \cdot n$  because  $tp(Y_i, \preceq) \leq \omega_1$ .

We first prove claim 3.3.1.

*Proof of claim 3.3.1.* Let  $A, B \in [n]^{<n}$  such that  $B \setminus A \neq \emptyset$ . Let  $k \in B \setminus A$ . Then  $Z_A \subset \overline{X_k}$  and  $Z_B \cap \overline{X_k} = \emptyset$ . Thus  $\overline{Z_A} \cap Z_B = \emptyset$ .  $\square$

We now prove claim 3.3.2.

*Proof of claim 3.3.2.* Since  $X \subset \bigcup \{Z_A \mid i \in n \wedge A \in [n]^i\}$ , and  $Z_A \cap Z_B = \emptyset$  for  $A \neq B$  by Claim 3.3.1, the family  $\{Y_i \mid i \in n\}$  forms a partition of  $X$ .

Fix  $i \in n$ . Then

$$\overline{\bigcup_{j \leq i} Y_j} \cap \left( \bigcup_{i < k < n} Y_k \right) = \bigcup_{j \leq i} \bigcup_{j < k < n} \bigcup_{A \in [n]^i} \bigcup_{B \in [n]^k} \overline{Z_A} \cap Z_B.$$

By 3.3.1 this set is empty. Thus  $\bigcup_{j \leq i} Y_j$  is closed in  $X$ .  $\square$

The plan for the proof of Claim 3.3.3 is that for each  $i \in n$  and each  $A \in [n]^i$  we will well order  $Z_A$ , shuffle  $\{Z_A \mid A \in [n]^i\}$  together and then show that this well ordering will witness that  $Y_i$  is left-separated and  $\text{ord}_\ell(Y_i) \leq \omega_1$ .

Let  $(N_\alpha)_{\alpha < \omega_1}$  be a continuous chain of countable elementary submodels such that  $X_i \in N_0$  for each  $i \in n$  and  $X \subset \bigcup_{\alpha < \omega_1} N_\alpha$ .

*Proof of Claim 3.3.3.* Fix  $i \in n$ . For each  $\alpha < \omega_1$  the set  $Y_i \cap N_{\alpha+1} \setminus N_\alpha$  is countable and we write it as an  $\omega$  sequence  $\{z(\alpha, j) \mid j \in \omega\}$ . Fix  $\alpha < \omega_1$ . Then

$$Y_i = \{z(\alpha, j) \mid j \in \omega \wedge \alpha < \omega_1\}$$

and using the lexicographic ordering this well orders  $Y_i$  in order type less than or equal to  $\omega_1$ . We now show that this well ordering witnesses that  $Y_i$  is left-separated.

Since  $Z_A \cap \overline{Z_B} = \emptyset$  for  $A \neq B$  with  $A, B \in [n]^i$  by Claim 3.3.1, it is enough to prove that every  $Z_A$  is left-separated in our well-ordering.

Fix  $A \in [n]^i$ .

Let  $z(\alpha, m) \in Z_A$ . Then  $z(\alpha, m) \in N_{\alpha+1} \setminus N_\alpha$ . To show that  $Z_A$  is left-separated it is sufficient to show that  $z(\alpha, m) \notin \overline{Z_A \cap N_\alpha}$ .

**Claim 3.3.4.** For all  $j \notin A$ ,  $\overline{Z_A \cap N_\alpha} \subset \overline{X_j \cap N_\alpha}$ .

*Proof.* Fix  $j \notin A$ . Let  $x \in Z_A \cap N_\alpha$ . So  $x \in \overline{X_j} \cap N_\alpha$ . Since  $t(X) = \omega$ , let  $H \in [X_j]^\omega \cap N_\alpha$  so that  $x \in \overline{H}$ . Since  $|H| = \omega$  and  $H \in N_\alpha$  we have that  $H \subset N_\alpha$ . Thus,  $H \subset X_j \cap N_\alpha$  and so  $x \in \overline{X_j \cap N_\alpha}$ .

Thus  $Z_A \cap N_\alpha \subset \overline{X_j \cap N_\alpha}$ , and so  $\overline{Z_A \cap N_\alpha} \subset \overline{X_j \cap N_\alpha}$ .  $\square$

**Claim 3.3.5.**  $\overline{Z_A \cap N_\alpha} \cap Z_A \subset N_\alpha$ .

*Proof.* Let  $x \in \overline{Z_A \cap N_\alpha} \cap Z_A$ . Then  $x \notin \bigcup_{j \in A} \overline{X_j}$ . Let  $j \in (n \setminus A)$  such that  $x \in X_j$ . Since  $x \in \overline{Z_A \cap N_\alpha}$  and  $j \notin A$ , by the previous claim we have that  $x \in \overline{X_j \cap N_\alpha} \cap X_j$ . As  $X_j$  is left-separated, this implies that  $x \in N_\alpha$ .  $\square$

Now let us return to  $z(\alpha, m)$ . Since  $z(\alpha, m) \notin N_\alpha$  and  $z(\alpha, m) \in Z_A$ , then we have that  $z(\alpha, m) \notin \overline{Z_A \cap N_\alpha}$  which completes the proof.  $\square$

$\square$

#### 4. “LEFT-SEPARATED” PROPERTY IN C.C.C. GENERIC EXTENSIONS

In this section we prove theorem 1.6.

To prove theorem 1.6(1) we need the following lemma:

**Lemma 4.1.** *Assume that  $X$  is a topological space,  $|X| = \omega_1$ ,  $Q$  is a c.c.c poset such that*

$$V^Q \models X = A \cup B, A \subset^{closed} X, \text{ord}_\ell(B) = \omega_1.$$

*Then there is a partition  $X = A^* \cup B^*$  in the ground model such that*

- (1)  $A^* \subset^{closed} X, \text{ord}_\ell(B^*) \leq \omega_1$ ,
- (2)  $1_Q \Vdash A^* \subset \dot{A}$ .

*Proof of lemma 4.1.* We can assume that the underlying set of  $X$  is  $\omega_1$ .

Assume that

$$1_Q \Vdash \dot{f} : \omega_1 \rightarrow \dot{B} \text{ is bijection witnessing } \text{ord}_\ell(\dot{B}) = \omega_1.$$

For  $\alpha \in \omega_1$ , let

$$b(\alpha) = \{x \in X : \exists q \in Q \ q \Vdash f(\alpha) = x\}.$$

Clearly  $b(\alpha) \in [X]^\omega$ . For  $b \in [X]^\omega$  let

$$\alpha_b = \sup\{\gamma < \omega_1 : \exists q \in Q \ q \Vdash \dot{f}(\gamma) \in b\}.$$

Let

$$E = \{\gamma < \omega_1 : (\forall \zeta < \gamma) \ \alpha_{b(\zeta)} < \gamma\}.$$

Then

$$1_Q \Vdash \dot{f}''\varepsilon = \varepsilon \cap \dot{B} \tag{4.1}$$

for all  $\varepsilon \in E$ .

**Claim 4.1.1.**  $E \subset \omega_1$  is club.

$E$  is clearly closed. If  $\delta < \omega_1$ , then define countable ordinals  $\delta_n$  and countable subsets  $D_n$  of  $\omega_1$  for  $n < \omega$  as follows:

- $\delta_0 = \delta$
- $D_n = \bigcup_{\beta < \delta_n} b(\beta)$ .
- $\delta_{n+1} = \alpha_{D_n}$ .

Then  $\gamma = \bigcup_{n < \omega} \delta_n \in E$  and  $\delta < \gamma$ . So  $E$  is unbounded as well, which proves the claim.  $\square$

By induction on  $\nu < \omega_2$ , we will define families

$$\{B_\xi^\nu : \xi < \omega_1\}$$

of pairwise disjoint countable subsets of  $\omega_1$  and subsets  $A^\nu \subset \omega_1$  as follows:

- Let  $\{\varepsilon_\xi : 1 \leq \xi < \omega_1\}$  be the increasing enumeration of  $E$ , write  $\varepsilon_0 = 0$ , and let

$$B_\xi^0 = \varepsilon_{\xi+1} \setminus \varepsilon_\xi$$

for  $\xi < \omega_1$ ;

- Let

$$A^\nu = X \setminus \bigcup_{\xi < \omega_1} B_\xi^\nu.$$

- If  $\nu$  is a limit ordinal, let

$$B_\xi^\nu = \bigcap_{\eta < \nu} B_\xi^\eta.$$

- if  $\nu = \mu + 1$ , let

$$B_\xi^{\mu+1} = B_\xi^\mu \setminus \overline{A^\mu \cup \bigcup_{\zeta < \xi} B_\zeta^\mu}.$$

**Claim 4.1.2.**  $1_Q \Vdash A^\nu \subset \dot{A}$ .

*Proof of the Claim.* By induction on  $\nu$ .

**Case 1.**  $\nu = 0$ .

Since  $A^0 = \emptyset$ , the statement is trivial.

**Case 2.**  $\nu$  is limit.

Then  $A^\nu = \bigcup_{\mu < \nu} A^\mu$ , so the statement is clear.

**Case 3.**  $\nu = \mu + 1$ .

Assume that  $x \in B_\xi^\mu \setminus B_\xi^{\mu+1}$ . Then

$$x \in \overline{A^\mu \cup \bigcup_{\zeta < \xi} B_\zeta^\mu}. \quad (4.2)$$

Moreover,

$$1_Q \Vdash A^\mu \subset \dot{A} \quad (4.3)$$

by the inductive assumption, and

$$\bigcup_{\zeta < \xi} B_\zeta^\mu \subset \bigcup_{\zeta < \xi} B_\zeta^0 = \varepsilon_\xi \quad (4.4)$$

by the construction. Putting together (4.1), (4.2), (4.3) and (4.4) we have

$$\begin{aligned} 1_Q \Vdash x \in \overline{\dot{A} \cup \varepsilon_\xi} &= \overline{\dot{A}} \cup \overline{\dot{B} \cap \varepsilon_\xi} = \overline{\dot{A}} \cup \overline{\dot{f}'' \varepsilon_\xi} = \\ &= \dot{A} \cup (\dot{f}'' \varepsilon_\xi) = \dot{A} \cup (\dot{B} \cap \varepsilon_\xi). \end{aligned}$$

Since  $x \notin \varepsilon_\xi$ , it follows that

$$1_Q \Vdash x \in \dot{A},$$

which completes the proof of the claim.  $\square$

For all  $\xi < \omega_1$ , the family  $\langle B_\xi^\nu : \nu < \omega_2 \rangle$  is decreasing, so there is  $\nu_\xi < \omega_2$  such that

$$B_\xi^\nu = B_\xi^{\nu_\xi}.$$

for all  $\nu_\xi \leq \nu < \omega_2$ . So there is  $\mu < \omega_2$  such that

$$B_\xi^{\mu+1} = B_\xi^\mu \text{ for all } \xi < \omega_1,$$

i.e.

$$A^\mu \cup \bigcup_{\zeta < \xi} B_\zeta^\mu \text{ is closed in } X \text{ for all } \xi < \omega_1 \quad (4.5)$$

Let

$$A^* = A^\mu \text{ and } B^* = X \setminus A^* = \bigcup_{\xi < \omega_1} B_\xi^\mu.$$

Then  $A^*$  is closed in  $X$  by (4.5). So  $\bigcup_{\zeta < \xi} B_\zeta^\mu$  is also closed in  $X \setminus A$  for all  $\xi < \omega_1$ . So  $B = X \setminus A$  is left separated in type  $\leq \omega_1$ : order every  $B_\xi^\mu$  is type  $\leq \omega$ , and declare that, for all  $\zeta < \xi$ , every element of  $B_\zeta^\mu$  is less than every element of  $B_\xi^\mu$ :

$$B_0^\mu < B_\mu^1 < \dots < B_\zeta^\mu < \dots < B_\xi^\mu < \dots$$

$\square$

*Proof of theorem 1.6(1).* By induction on  $n \in \omega$  we prove the following statement:

( $*_n$ ) If  $X$  is a topological space,  $Q$  is a c.c.c poset such that

$$V^Q \models \text{ord}_\ell(X) \leq \omega_1 \cdot n$$

then  $X$  is left-separated in even in  $V$ , moreover

$$\text{ord}_\ell(X)^V \leq \omega_1 \cdot n.$$



Assume that  $(*_m)$  holds for  $m < n$ , and  $X$  is a topological space,  $Q$  is a c.c.c poset such that

$$V^Q \models \text{ord}_\ell(X) \leq \omega_1 \cdot n$$

We can assume that in  $V^Q$  the space  $X$  has a left-separating well-ordering in type  $\omega_1 \cdot n$ . (If  $\text{ord}_\ell(X) < \omega_1 \cdot n$ , add  $\omega_1$  isolated points to  $X$ ). Let  $B$  be the last  $\omega_1$  many points of  $X$  in that well-ordering and  $A = X \setminus B$ .

By lemma 4.1 then there is a partition  $X = C \cup D$  in the ground model such that

- (1)  $C \subset^{closed} X$ ,  $\text{ord}_\ell(D) \leq \omega_1$ ,
- (2)  $1_Q \Vdash C \subset A$ .

Then

$$V^Q \models \text{ord}_\ell(C) \leq \text{ord}_\ell(A) \leq \omega_1 \cdot (n - 1),$$

so by the inductive assumption, we have  $\text{ord}_\ell(C) \leq \omega_1 \cdot (n - 1)$  in the ground model. Thus  $\text{ord}_\ell(X) \leq \text{ord}_\ell(C) + \text{ord}_\ell(D) = \omega_1 \cdot (n - 1) + \omega_1 = \omega_1 \cdot n$  because  $C$  is closed in  $X$ . So we proved theorem 1.6(1).  $\square$

Next we show that, consistently, a non-left-separated space may be left-separated in some c.c.c generic extension of the ground model. The construction of the model of theorem 4.4 is quite complicated, so first we consider an other example which is much simpler, but also much weaker.

Let us recall that an uncountable subset of the reals is called *Luzin* if it has countable intersection with every meager set. Mahlo and Luzin, independently, proved that if CH holds, then there is a Luzin set.

**Theorem 4.2.** *If there is a Luzin set, then there is a Hausdorff topological space  $X$  such that  $\text{ord}_\ell(X) = \infty$ , but  $\text{ord}_\ell(X) = \omega_1 \cdot \omega$  is some c.c.c generic extension.*

*Proof.* Since there is a Luzin set  $Z$ , there is a Luzin  $Y$  such that  $|Z \cap V| = \omega_1$  for all non-empty euclidean open set  $V$ . We can assume that  $Y$  can not contain any rational numbers.

The underlying set of our space is  $X = \mathbb{Q} \cup Y$ , and the topology  $\tau$  on  $X$  is the refinement of the euclidean topology by declaring the countable subsets of  $Y$  to be closed. So the following family  $\mathcal{B}$  is a base of  $X$ :

$$\mathcal{B} = \{U \setminus Z : U \text{ is a euclidean open set}, Z \in [Y]^\omega\}.$$

Assume that  $F \subset Y$  is nowhere dense in the topology  $\tau$ . Then there is euclidean dense open set  $U$  and a countable subset  $Z$  of  $Y$  such that

$F \cap (U \setminus Z) = \emptyset$ . So  $F \cap U \subset Z$ . Since  $|F \setminus U| \leq \omega$  because  $Y$  is Luzin, we have  $|F| \leq \omega$ .

Since the countable subsets of  $Y$  are closed and  $\Delta(Y) = \omega_1$ , it follows that  $F \subset Y$  is nowhere dense in  $\tau$  iff  $Y$  is countable.

So we can apply lemma 3.1 to show that  $X$  is not left-separated: let  $A = \mathbb{Q}$  and  $B = Y$ . Since the union of countably many nowhere dense subset of  $Y$  is nowhere dense, we have that  $X$  is not left separated.

Consider a c.c.c generic extension  $W$  of the ground model in which Martin's Axiom holds. Then in  $W$  we have countable many dense open sets  $\{U_n : n \in \omega\}$  such that

$$\mathbb{Q} \subset \bigcap_{n \in \omega} U_n \subset \mathbb{R} \setminus Y.$$

Let

$$F_n = (Y \setminus \bigcap_{i \in n} U_i) \setminus \bigcup_{m < n} F_m.$$

Then  $F_n$  is a closed subset of  $X$  and it is left-separated in type  $\omega_1$  because the countable subsets of  $Y$  are all closed. Write  $\mathbb{Q} = \{q_n : n \in \omega\}$ . Then we have a left-separating order of  $X$  in type  $\omega_1 \cdot \omega$ :

$$F_0 < q_0 < F_1 < q_1 < F_1 < \dots$$

where  $F_n$  is ordered in type  $\omega_1$ . □

Instead of theorem 1.6(2) we will prove theorem 4.4, which is a stronger statement. To formulate that theorem. we need some preparation.

First we recall a definition from the proof of [5, Theorem 3.5].

**Definition 4.3.** For each uncountable cardinal  $\kappa$  define the poset  $\mathcal{P}^\kappa = \langle P^\kappa, \leq \rangle$  as follows.

A quadruple  $\langle A, n, f, g \rangle$  is in  $P_0^\kappa$  provided (1)–(5) below hold:

- (1)  $A \in [\kappa]^{<\omega}$ ,
- (2)  $n \in \omega$ ,
- (3)  $f$  and  $g$  are functions,
- (4)  $f : A \times A \times n \rightarrow 2$ ,
- (5)  $g : A \times n \times A \times n \rightarrow 3$ ,

For  $p \in P_0^\kappa$  we write  $p = \langle A^p, n^p, f^p, g^p \rangle$ . If  $p, q \in P_0^\kappa$  we set

$$p \leq q \text{ iff } f^p \supseteq f^q \text{ and } g^p \supseteq g^q.$$

If  $p \in P_0^\kappa$ ,  $\alpha \in A^p$ ,  $i < n^p$  set

$$U(\alpha, i) = U^p(\alpha, i) = \{\beta \in A^p : f^p(\beta, \alpha, i) = 1\}.$$

A quadruple  $\langle A, n, f, g \rangle \in P_0^\kappa$  is in  $P^\kappa$  iff (i)–(iv) below are also satisfied:

- (i)  $\forall \alpha \in A \forall i < n \alpha \in U(\alpha, i)$ ,

- (ii)  $\forall \alpha \in A \forall i < j < n \ U(\alpha, j) \subset U(\alpha, i),$
- (iii)  $\forall \{\alpha, \beta\} \in [A]^2 \forall i, j < n$   
 $U(\alpha, i) \subset U(\beta, j) \iff g(\alpha, i, \beta, j) = 0,$   
 $U(\alpha, i) \cap U(\beta, j) = \emptyset \iff g(\alpha, i, \beta, j) = 1.$
- (iv)  $\forall \{\alpha, \beta\} \in [A]^2 \forall i, j < n$   
 if  $\alpha \in U(\beta, j)$  and  $\beta \in U(\alpha, i)$  then  $g(\alpha, i, \beta, j) = 2.$

Finally let  $\mathcal{P}^\kappa = \langle P^\kappa, \leq \rangle$ .

Using this definition we can formulate our next result:

**Theorem 4.4.** *For each uncountable cardinal  $\kappa$ , the poset  $V^{\mathcal{P}^\kappa}$  satisfies c.c.c and in  $V^{\mathcal{P}^\kappa}$  there is a 0-dimensional, first countable topological space  $X = \langle \kappa, \tau \rangle$  and there is a c.c.c posets  $Q$  satisfying the following conditions:*

- (a)  $V^{\mathcal{P}^\kappa} \models "R(X) = \omega, \text{ so } X \text{ does not contain uncountable left-separated subspaces}."$
- (b)  $V^{\mathcal{P}^\kappa * Q} \models "X \text{ is left separated in type } \kappa \cdot \omega"$ ,

*Proof of theorem 4.4.* It was proved in the proof of [5, Theorem 3.5] that the poset  $V^{\mathcal{P}^\kappa}$  satisfies c.c.c.

Let  $\mathcal{G}$  be the  $\mathcal{P}^\kappa$  generic filter and let  $F = \bigcup \{f^p : p \in \mathcal{G}\}$ . For each  $\alpha < \kappa$  and  $n \in \omega$  let  $V(\alpha, i) = \{\beta < \kappa : F(\beta, \alpha, i) = 1\}$ . Put  $\mathcal{B}_\alpha = \{V(\alpha, i) : i < \kappa\}$  and  $\mathcal{B} = \bigcup \{\mathcal{B}_\alpha : \alpha < \kappa\}$ . By standard density arguments we can see that  $\mathcal{B}$  is base of a first countable, 0-dimensional  $T_2$  space  $X = \langle \kappa, \tau \rangle$ .

[5, Theorem 3.5] claimed that  $R(X) = \omega$ , so  $X$  can not contain uncountable left-separated subspaces.

To prove that  $X$  is left separated in type  $\kappa \cdot \omega$  in some c.c.c generic extension  $V^{\mathcal{P}^\kappa} * Q$  define the poset  $Q$  in  $V^{\mathcal{P}^\kappa}$  as follows:

A triple  $\langle B, d, e \rangle$  is in  $Q$  iff

- (a)  $B \in [\kappa]^{<\omega},$
- (b)  $d : B \rightarrow \omega,$
- (c)  $e : B \rightarrow \omega,$
- (d) for each  $\{\alpha, \beta\} \in [B]^2$  if  $d(\alpha) \leq d(\beta)$  then  $\alpha \notin V(\beta, e(\beta)).$

The orderings on  $Q$  is defined in the straightforward way,

$$\langle B^0, d^0, e^0 \rangle \leq \langle B^1, d^1, e^1 \rangle \text{ iff } d^0 \supset d^1 \text{ and } e^0 \supset e^1.$$

If  $q$  and  $r$  are compatible elements of  $Q$ , then denote by  $q \wedge r$  their greatest lower bound in  $Q$ .

**Lemma 4.5.**  $\mathcal{P}^\kappa * Q$  satisfies c.c.c.

To prove this lemma we need to recall two more definitions and a lemma from [5].

**Definition 4.6** ([5, Definition 3.6]). Assume that  $p_i = \langle A^i, n^i, f^i, g^i \rangle \in P_0^\kappa$  for  $i \in 2$ . We say that  $p_0$  and  $p_1$  are *twins* iff  $n^0 = n^1$ ,  $|A^0| = |A^1|$  and taking  $n = n^0$  and denoting by  $\sigma$  the unique  $<$ -preserving bijection between  $A^0$  and  $A^1$  we have

- (i)  $\sigma \upharpoonright A^0 \cap A^1 = \text{id}_{A^0 \cap A^1}$ .
- (ii)  $\sigma$  is an isomorphism between  $p_0$  and  $p_1$ , i.e.  $\forall \alpha, \beta \in A^0, \forall i, j < n$ 
  - (ii-a)  $f^0(\alpha, \beta, i) = f^1(\sigma(\alpha), \sigma(\beta), i)$ ,
  - (ii-b)  $g^0(\alpha, i, \beta, j) = g^1(\sigma(\alpha), i, \sigma(\beta), j)$ ,

We say that  $\sigma$  is the *twin function* of  $p_0$  and  $p_1$ . Define the *smashing function*  $\bar{\sigma}$  of  $p_0$  and  $p_1$  as follows:  $\bar{\sigma} = \sigma \cup \text{id}_{A_1}$ . The function  $\sigma^*$  defined by the formula  $\sigma^* = \sigma \cup \sigma^{-1} \upharpoonright A_1$  is called the *exchange function* of  $p_0$  and  $p_1$ .

**Definition 4.7.** Assume that  $p_0$  and  $p_1$  are twins and  $\varepsilon : A^{p_1} \setminus A^{p_0} \rightarrow 2$ . A common extension  $q \in P^\kappa$  of  $p_0$  and  $p_1$  is called an  $\varepsilon$ -*amalgamation* of the twins provided

$$\forall \alpha \in A^{p_0} \triangle A^{p_1} \quad f^q(\alpha, \sigma^*(\alpha), i) = \varepsilon(\bar{\sigma}(\alpha)). \quad (4.6)$$

The notion of an  $\varepsilon$ -amalgamation was introduced in [5, Definition 3.7].

An  $\varepsilon$ -amalgamation is a *strong  $\varepsilon$ -amalgamation* if

$$\begin{aligned} \forall \{\alpha, \beta\} \in [A^{p_0} \cup A^{p_1}]^2 \quad \forall i < n^{p_0} \\ \text{if } \sigma^*(\alpha) \neq \sigma^*(\beta) \text{ then } f^q(\alpha, \beta, i) = f^\mu(\sigma^*(\alpha), \sigma^*(\beta), i). \quad (*) \end{aligned}$$

**Lemma 4.8.** *If  $p_0, p_1 \in \mathcal{P}^\kappa$  are twins and  $\varepsilon : A^{p_1} \setminus A^{p_0} \rightarrow 2$ , then  $p_0$  and  $p_1$  have a strong  $\varepsilon$ -amalgamation in  $P^\kappa$  such that .*

*Proof.* In [5, Lemma 3.8] we proved that  $p_0$  and  $p_1$  have an  $\varepsilon$ -amalgamation  $q$  in  $P^\kappa$ . However, the condition  $q$ , which was defined in the first paragraph of the proof of [5, Lemma 3.8] is actually a strong  $\varepsilon$ -amalgamation.  $\square$

Now we are ready to prove our lemma.

*Proof of Lemma 4.5.* Let  $\langle \langle p_\nu, q_\nu \rangle : \nu < \omega_1 \rangle \subset \mathcal{P}^\kappa * Q$ . We can assume that  $p_\nu$  decides  $q_\nu$ . Write  $p_\nu = \langle A^\nu, n^\nu, f^\nu, g^\nu \rangle$  and  $q_\nu = \langle B^\nu, d^\nu, e^\nu \rangle$ . By standard density arguments we can assume that  $A^\nu \supset B^\nu$ . Applying standard  $\Delta$ -system and counting arguments we can find  $\{\nu, \mu\} \in [\omega_1]^2$  such that  $p_\nu$  and  $p_\mu$  are twins and denoting by  $\sigma$  the twin function of  $p_\nu$  and  $p_\mu$  we have

- (iii)  $B^\mu = \sigma'' B^\nu$ ,
- (iv)  $d^\nu(\alpha) = d^\mu(\sigma(\alpha))$  for each  $\alpha \in B^\nu$ , and
- (v)  $e^\nu(\alpha) = e^\mu(\sigma(\alpha))$  for each  $\alpha \in B^\nu$ .

Define the function  $\varepsilon^0 : A^\mu \setminus A^\nu \rightarrow 2$  by the equation  $\varepsilon^0(\alpha) = 0$ . By Lemma 4.8 the conditions  $p^\nu$  and  $p^\mu$  have an  $\varepsilon^0$ -amalgamation  $p$ . We claim that

$$p \Vdash q_\nu \text{ and } q_\mu \text{ are compatible in } Q,$$

i.e.

$$p \Vdash \langle B^\nu \cup B^\mu, d^\nu \cup d^\mu, e^\mu \cup e^\mu \rangle \text{ has properties (a)–(d).}$$

(a) clearly holds. Since  $\sigma(\alpha) = \alpha$  for each  $\alpha \in B^\nu \cap B^\mu \subset A^\nu \cap A^\mu$ , assumption (iv) implies that  $d^\nu \cup d^\mu$  is a function, and assumption (v) implies that  $e^\nu \cup e^\mu$  is a function, and so (b) and (d) also hold.

To check (d) it is enough to show that if  $\alpha \in B^\nu$ ,  $\beta \in B^\mu$ ,  $d^\nu(\alpha) \leq d^\mu(\beta)$ , then  $p \Vdash \alpha \notin V(\beta, e^\mu(\beta))$ , i.e.

$$f^p(\alpha, \beta, e^\mu(\beta)) = 0. \quad (\star)$$

Assume first that  $\sigma(\alpha) = \sigma^*(\alpha) \neq \beta$ . Then

$$d = d^\mu(\sigma(\alpha)) = d^\nu(\alpha)$$

by (iv),

$$e = e^\mu(\sigma(\alpha)) = e^\nu(\alpha)$$

by (v), and so  $d^\mu(\sigma(\alpha)) \leq d^\mu(\beta)$ . Thus, by (d),

$$p_\mu \Vdash \sigma(\alpha) \notin V(\beta, e^\mu(\beta)),$$

i.e.

$$f^\mu(\sigma(\alpha), \beta, e^\mu(\beta)) = 0.$$

Since  $p$  is a strong  $\varepsilon^0$ -amalgamation, condition  $(*)$  implies that

$$f^p(\alpha, \beta, e^\mu(\beta)) = f^\mu(\sigma(\alpha), \beta, e^\mu(\beta)) = 0,$$

i.e.  $(\star)$  holds.

If  $\sigma^*(\alpha) = \beta$ , then  $p \Vdash \alpha \notin V(\beta, e^\mu(\beta))$  because  $p$  is an  $\varepsilon^0$ -amalgamation, and so

$$f^p(\alpha, \sigma^*(\alpha), e^\mu(\beta)) = \varepsilon(\overline{\sigma}(\alpha)) = 0.$$

□

**Lemma 4.9.**

$$V^{\mathcal{P}^\kappa * Q} \models \text{“}X \text{ is left-separated in type } \kappa \cdot \omega\text{”}$$

*Proof.* Let  $\mathcal{H}$  be the  $Q$ -generic filter over  $V^{\mathcal{P}^\kappa}$ . Set  $d = \bigcup \{d^q : q \in \mathcal{H}\}$  and  $e = \bigcup \{e^q : q \in \mathcal{H}\}$ . By standard density arguments the domains of the functions  $d$  and  $e$  are  $\kappa$ . We have

$$V(x, e(x)) \cap \left( \bigcup_{i \leq d(x)} d^{-1}\{i\} \right) = \{x\}, \quad (4.7)$$

by (d), so  $d^{-1}(n)$  is discrete and  $\overline{d^{-1}\{n\}} \cap d^{-1}n = \emptyset$  for each  $n \in \omega$ . So we can construct a left-separating ordering of  $X$  in type  $\kappa \cdot \omega$  as follows:

consider a well-ordering  $\prec$  of  $X$  such that  $d^{-1}\{n\}$  is well-ordered in type  $\kappa$  for  $n \in \omega$ , and

$$d^{-1}\{0\} \prec d^{-1}\{1\} \prec \dots < d^{-1}\{n\} \prec d^{-1}\{n+1\} \prec \dots$$

□

This completes the proof of theorem 4.4.

□

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